Exponential of Derivative is Shift

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Recall the Taylor series for e^x around 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let's plug in $a\frac{\mathrm{d}}{\mathrm{d}t}$, and apply it to an analytic function $f : \mathbb{R} \to \mathbb{R}$.

$$e^{a\frac{\mathrm{d}}{\mathrm{d}t}} f = \sum_{n=0}^{\infty} \frac{1}{n!} \left(a^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \right) f = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}$$

Now more generally, the formula for a Taylor series of f around t is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{n!} (x - t)^n$$

so if we let $x = t + a$, then this expression becomes

$$f(t + a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{n!} ((t + a) - t)^n = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(t)$$

which is exactly $e^{a\frac{\mathrm{d}}{\mathrm{d}t}} f$ evaluated at t. Hence, $e^{a\frac{\mathrm{d}}{\mathrm{d}t}} f(t) = f(t + a)$.