Properties of exp

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When the power series $\sum_{n=0}^{\infty} \frac{X^n}{n!}$ is well-defined (including when X is a real number, complex number, or matrix), it is equal to $\exp(X)$. This truth is so ubiquitous that it is often used as a definition of exp. Here, I provide some more precise arguments on this subject.

1. $\exp(tX) = y(t)$

   As mentioned in a footnote, I wrote that $\exp(tX) = y(t)$ in the definition of exp, but this shouldn't be part of the definition. Rather, it follows from the definition $\exp(X) = y(1)$. To see why, define $z(s) = y(ts)$. Then $\exp(tX) = z(1) = y(t)$, so we just need to show the tangent vector $\frac{\mathrm{d}z}{\mathrm{d}s}(0)$ is tX

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\mathrm{d}z}{\mathrm{d}(ts)}\frac{\mathrm{d}(ts)}{\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot t$$

   which, evaluated at 0, gives tX.

2. $\exp(X)\exp(Y) = \exp(X+Y)$

   This one isn't always true, in fact it's true only when X and Y commute, i.e. $XY = YX$. We can prove it by matching terms of the power series. Computing the degree-n term of $\exp(X)\exp(Y)$ involves multiplying all terms with degrees that sum to n (since exponents are additive), so

$$\exp(X)\exp(Y) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \sum_{\ell=0}^{\infty} \frac{Y^\ell}{\ell!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{X^k Y^{n-k}}{k!(n-k)!}$$

   Recalling the binomial theorem, we may conclude

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{X^k Y^{n-k}}{k!(n-k)!} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!} = \exp(X+Y)\end{aligned}$$

   It immediately follows that $\exp(0) = 1$ by considering $\exp(0) = \exp(0+0)$.

3. $\exp(X) = \lim_{n\to\infty}(1+\frac{X}{n})^n$

The binomial theorem tells us that the coefficient of X^k in $(1+\frac{X}{n})^n$ is $\binom{n}{k}/n^k$. The coefficient of X^k in $\exp(X)$ is $\frac{1}{k!}$, so we just need to show that

$$\lim_{n\to\infty}\frac{\binom{n}{k}}{n^k} = \frac{1}{k!}, \text{ or } \lim_{n\to\infty}\frac{k!\binom{n}{k}}{n^k} = 1$$

Observe that the above expression simplifies to

$$\lim_{n\to\infty}\frac{\frac{n!}{(n-k)!}}{n^k} = \lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{n^k} = \lim_{n\to\infty}\prod_{\ell=0}^{k-1}\frac{n-\ell}{n}$$

Then, it's a straight shot to the finish line

$$\lim_{n\to\infty}\prod_{\ell=0}^{k-1}\frac{n-\ell}{n} = \prod_{\ell=0}^{k-1}\lim_{n\to\infty}\frac{n-\ell}{n} = \prod_{\ell=0}^{k-1}1 = 1$$

4. $e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$

e^x is only defined for $x\in\mathbb{C}$, so we can leverage properties of complex numbers not present in matrices, namely division.

From the previous proof, we know $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$. Using exponent rules, we have

$$e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$$

Let $k = nx$, so $k\to\infty$ when $n\to\infty$. Then

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = \lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k$$

Combining this result with the previous one, we have that $e^x = \exp(x)$ for $x\in\mathbb{C}$, as we expect.