Continuity of Measure

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1 Introduction to Measure

For suitable $A \subseteq \mathbb{R}$, call $\mu(A)$ the *measure* of A, which roughly corresponds to "size". The details of how we define it aren't important here, but its properties are mostly intuitive. In particular, $\mu([a, b]) = b - a$ and for pairwise disjoint sets $A_n \subseteq \mathbb{R}$, we have $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$.

There are even sets with infinite measure, like $(0, \infty)$ or even $\mathbb{R}$ itself, to which we assign $\mu(\mathbb{R}) = \infty$. If it helps to see the "type signature", μ's is given by $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ where $\mathbb{R}_{\geq 0}$ is the nonnegative reals and $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$ is the set of "measurable" subsets of $\mathbb{R}$.

As you have probably learned in your analysis course by now, we must take care in exchanging limits with other operations. Does the measure of an infinite intersection equal the limit of the measure of each set? What about union? (Clearly not, let $A_n = (n, n + 1)$. But there's still something we can say.)

2 Continuity from Below

Let $A_n \subseteq \mathbb{R}$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, so A_n is an increasing sequence of sets. For example, let $A_n = (0, 1 - \frac{1}{n})$. Then $\mu(\bigcup_n A_n) = \lim_n \mu(A_n)$. Intuitively, each set A_n contains the previous set A_{n-1}, so their infinite union contains all of them, and is the "increasing limit" of the sets in a certain sense. Note that this increasing condition is why our example in the introduction didn't work.

To prove this, let $B_n = A_n \setminus A_{n-1}$ (with $A_0 = \emptyset$). It can be helpful to think of each B_n as the "layer" of an onion, with the union $\bigcup_n A_n = \bigcup_n B_n$ being the whole onion. Take note that by defining B_n by set differences, each B_n is disjoint, and we have turned the union $\bigcup_n A_n$ into the disjoint union $\bigcup_n B_n$. Then,

$$\mu\left(\bigcup_n A_n\right) = \mu\left(\bigcup_n B_n\right) = \sum_n \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu(A_n)$$

Crucially, the second equality is valid because the B_n are pairwise disjoint.

3 Continuity from Above

One might understandably expect a similar condition on intersections. Namely, for sets $A_n \subseteq \mathbb{R}$ with $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, we hope for something like $\mu(\bigcap_n A_n) = \lim_n \mu(A_n)$. And that's *almost* true – but there's a complicating factor, sets of infinite measure. For instance, consider $A_n = (n, \infty)$. Then $\mu(A_n) = \infty$ for each A_n, but $\bigcap_n A_n = \emptyset$, which has measure zero.

So we impose the additional condition that $\mu(A_n) < \infty$ for some A_n and thus all the A_n after it. Now, in fact, our theorem holds! But the proof for this one is harder. You can make the simplifying assumption that $\mu(A_1) < \infty$, since we'll eventually come to the set of finite measure anyway. It's also easier to prove the case where $\mu(\bigcap_n A_n) = \emptyset$.

4 Why "continuity"?

Why do we call it "continuity"? The usual definition of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ is that $\lim_{x\to a} f(x) = f(a)$ for all $a \in \mathbb{R}$. However, there's an alternative but equivalent condition called *sequential continuity*: for any sequence (x_n) such that $x_n \to a$, we have $f(x_n) \to f(a)$, for all $a \in \mathbb{R}$. In other words, f maps convergent sequences to convergent sequences.

How is this related? Well, if A_n is an increasing sequence of sets, then $\bigcup_n A_n$ is the "limit" of A_n in a certain sense (this analogy breaks down when A_n is not increasing); in other words, A_n "converges" to $\bigcup_n A_n$. So for μ to be "continuous", we would want it to map the convergent sequence A_n to a convergent sequence of reals, like with sequential continuity. And indeed, $\mu(\bigcup_n A_n) = \lim_n \mu(A_n)$. The same idea applies to continuity from above, of course.